(Chapter 3)
Matrices, Determinants and linear equations

A matrix is a rectangular array of numbers written within brackets. Each number in matrix is called an element of the matrix, such as the matrix A in the form :

$$A = \begin{array}{l} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{array}{l} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Column1} & \text{Column2} & \text{Column3} \end{array} \end{array}$$

where a_{11}, a_{12}, a_{13}, a_{21}, ... are the elements of matrix A and the elements (a_{11}, a_{22}, a_{33}) are the diagonal elements.

If a matrix has m rows and n columns it is called an $(m \times n)$ matrix. The expression $m \times n$ is called the size of the matrix and the numbers m and n are called the dimensions of the matrix.

The matrix with n rows and n columns is called square matrix of order n.

The matrix with only one column is called a column matrix and the matrix with only one row is called a row matrix.

properties of matrices:

1. The sum of two matrices of the same size, A and B denoted by $A+B$ is the matrix with elements that are the sums of the corresponding elements of A and B.

If $A = (a_{ij})_{m \times n}$

and $B = (b_{ij})_{m \times n}$

Then $C = A + B = (C_{ij})_{m \times n}$

$$C_{ij} = a_{ij} + b_{ij}$$

a. Note that the addition is not defined for matrices of different sizes or orders.

b. $A + B = B + A$

c. If A, B and C are three matrices with the same size then:

$$(A+B)+C = A + (B+C)$$

d- The negative of matrix M is denoted by $-M$ and it is the matrix with elements that are negative of the elements of M, therefore:

$$M + (-M) = 0$$

where 0 is a matrix which all its elements are equal zero.

e- If A, B are matrices with the same size, then:

$$A - B = A + (-B)$$

2. Multiplication

a. Multiplication of matrix by a number

If A is a matrix and α is a number then if we multiply A by α, we get a matrix whose elements are the same elements of the original matrix multiplied by the number α.

Example: If $A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$

Then $3A = \begin{pmatrix} 6 & 3 \\ 12 & 0 \end{pmatrix}$

b. Multiplication of Row matrix by Column matrix.

If R is a row matrix defined by:

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \end{pmatrix}$$

And if D is a column matrix defined by:

$$D = \begin{pmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{pmatrix}$$

Then $RD = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \end{pmatrix} \times \begin{pmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{pmatrix}$

$$= r_{11} d_{11} + r_{12} d_{21} + \cdots + r_{1n} d_{n1}$$

$$= (1 \times 1) \text{ matrix } = \text{ pure number}$$

Example: If $R = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and

$$D = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

Then $RD = 1 \times 2 + 2 \times 0 + 3 \times 4$
$= 14$

Note that if we consider R is

a vector and D is another vector then RD is represented by the dot product (scalar product) of R and D (R.D).

c. Multiplication of two matrices:

If A is a matrix of size $m \times p$ and B is a matrix of size $p \times n$

(number of columns in A is equal to the number of rows in B)

Then the multiplication AB is defined as the matrix C with size $m \times n$ (no. of rows in A × no. of columns in B) whose elements can be calculated by:

$$B = (b_{ij})_{p \times n} \quad , \quad A = (a_{ij})_{m \times p}$$

$$AB = C = (C_{ij})_{m \times n}$$

where $$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Note that if the number of columns in the first matrix is not equal to the number of rows in the second matrix, then the multiplication is not defined.

Example :

If $A = \begin{pmatrix} 1 & 5 \\ -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ 1 & 2 \\ 3 & -2 \end{pmatrix}$

Calculate AB and BA.

solution :

AB is not defined because the number of columns in A is 2 and the number of rows in B is 3 (not equal).

$$BA = \begin{pmatrix} 0 & 5 \\ 1 & 2 \\ 3 & -2 \end{pmatrix}_{3\times 2} \cdot \begin{pmatrix} 1 & 5 \\ -2 & 4 \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} 0\times 1 + 5\times -2 & 0\times 5 + 5\times 4 \\ 1\times 1 + 2\times -2 & 1\times 5 + 2\times 4 \\ 3\times 1 - 2\times -2 & 3\times 5 - 2\times 4 \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} -10 & 20 \\ -3 & 13 \\ 7 & 7 \end{pmatrix}$$

properties of matrix multiplications :

1. $AB \neq BA$
2. $\alpha (A + B) = \alpha A + \alpha B$
3. $A (B + C) = AB + AC$
4. $(AB)C = A(BC)$

Where α is a number
and A, B and C are matrices.

* Types of Matrices :

1. Unit matrix (identity matrix) المصفوفة الوحدة

It is a square matrix of order n whose all diagonal elements are equal to 1 and all other elements are equal to zero.

∴ The unit matrix of order 2 is:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And the unit matrix of order 3 is:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that if A is a matrix of size $n \times m$ then:

$$I_n A = A I_m = A$$

prove that !!

2. Upper triangle matrix

It is a matrix whose all of its elements located below the diagonal are equal to zero.

Example :

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Transpose Matrix

The transpose matrix of $A_{m \times n}$ is a matrix with size $n \times m$ and defined by A^T. We can get the elements of A^T by interchanging the rows of the original matrix by its columns.

Example :

If $A = \begin{pmatrix} -1 & 3 & 0 & 2 \\ 1 & 1 & -4 & 6 \end{pmatrix}_{2 \times 4}$

Then $A^T = \begin{pmatrix} -1 & 1 \\ 3 & 1 \\ 0 & -4 \\ 2 & 6 \end{pmatrix}_{4 \times 2}$

Some properties of transpose matrix:

a. $I_n = (I_n)^T$ wher I_n is the unit matrix of order n.

b. If A and B are two matrices then :

$$(AB)^T = B^T A^T$$

c. If A is a matrix, then:

$$(A^T)^T = A$$

Home Work :

If $B = \begin{pmatrix} 0 & 5 \\ 1 & 2 \\ 3 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

prove that $(BC)^T = C^T B^T$

4. Inverse Matrix :

The inverse matrix of a square matrix A of order n is a matrix of the same order such that :

$$A^{-1} A = A A^{-1} = I_n$$

Where A^{-1} is the inverse matrix of A and I_n is the unit matrix with same order of A.

The inverse matrix is defined for square matrices only.

5. Hermitian matrix :

If A is an $m \times n$ complex numbers matrix then :

a. The conjugate matrix of A is called A^* and can be calculated by taking the conjugates of the complex elements of A

b. The transpose of the conjugate matrix is called A^+

$$A^+ = (A^*)^T$$

c. The matrix A is Hermitian if :

$$A^+ = A$$

Example :

$$\text{If} \quad A = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$$

$$\text{Then} \quad A^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 3 \end{pmatrix}$$

$$\therefore \quad (A^*)^T = A^+ = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$$

$\therefore A = A^+$ $\quad$ $\therefore$ A is Hermitian matrix

6. Unitary Matrix : مصفوفة احادية

The square matrix A is unitary if :

$$A A^{\dagger} = I$$

By multiplying with A^{-1} :

$$A^{-1} A A^{\dagger} = A^{-1} I$$

$$\therefore \quad A^{\dagger} = A^{-1}$$

Example :

The matrix $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$

is unitary because :

$$A^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

and $\quad A A^{\dagger} = I_2 \quad$ (prove that)

7. Orthogonal Matrix :

The matrix A is orthogonal if :

$$A^{T} = A^{-1}$$

$$A A^{T} = I$$

Example :

The unit matrix : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

is orthogonal because :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Determinants : المحددات

Determinant is an arrangement of $(n \times n)$ elements distributed in n rows and n columns where n is called the order of determinant.

* The determinant is equal to a number.
* The determinant of A is denoted by (det A) or $|A|$.

A second-order determinant is written as :

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Its value is equal to :

$$a_{11} a_{22} - a_{12} a_{21}$$

A third-order determinant is written as:

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The n-order determinant is written as:

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

To evaluate the value of the determinant of order $n \geq 3$ we will define the co-factors of the determinant.

To explain the idea of the co-factor we will take a 3-order determinant:

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The co-factor of any element in the determinant is defined as:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Class

Where M_{ij} is the minor determinant of the element a_{ij} which can be obtained by evaluating the determinant remained after deleting the row and column that contain the element a_{ij}

Example:

Find the co-factors of a_{11} and a_{32} for the determinant

$$A = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 5 & 1 \\ 2 & 1 & 7 \end{vmatrix}$$

Solution:

* $C_{11} = (-1)^{1+1} \cdot M_{11}$

$$= + \begin{vmatrix} 5 & 1 \\ 1 & 7 \end{vmatrix} = 34$$

* $C_{32} = (-1)^{3+2} \cdot M_{32}$

$$= - \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

The value of any determinant can be found using the formula:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$ where i is constant (chosen row)

or

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$$ where j is constant (chosen column)

Example :

Evaluate the value of the determinant :

$$|A| = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 5 & 2 \\ 4 & 1 & 3 \end{vmatrix}$$

By using the second row first and by using the third column.

Solutions :

1. The minors : M_{21} , M_{22} . M_{23}

$$M_{21} = \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 11$$

$$M_{22} = \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = 5.$$

$$M_{23} = \begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} = -13$$

Class

The co-factors C_{21} , C_{22} , C_{23}

$C_{21} = (-1)^{2+1} \cdot M_{21} = -11$

$C_{22} = (-1)^{2+2} \cdot M_{22} = 5$

$C_{23} = (-1)^{2+3} \cdot M_{23} = 13$

$\therefore \quad |A| = a_{21} \cdot C_{21} + a_{22} \cdot C_{22} + a_{23} \cdot C_{23}$

$= 2 * (-11) + 5 * 5 + 2 * 13$

$= 29$

2. By using Column number 3 :

We can find :

$$\left.\begin{aligned} C_{13} &= -18 \\ C_{23} &= 13 \\ C_{33} &= 7 \end{aligned}\right\} \text{Home work}$$

$\therefore \quad |A| = a_{13} \cdot C_{13} + a_{23} \cdot C_{23} + a_{33} \cdot C_{33}$

$= 1 * (-18) + 2 * 13 + 3 * 7$

$= 29$

Properties of Determinants :

1. The value of determinant is equal to zero if :

   a- All the elements of one of the rows or columns are equal to zero.

   b- The corresponding elements of any two rows or columns are equal.

   c- The corresponding elements of any two rows or columns are proportional.

2. If any two rows or columns of determinant are interchanged, then the value of the new determinant is the negative of the original one.

3. If each element of one of the rows or columns is multiplied by Constant (K), then the value of the new determinant is equal to K times the value of the original one.

4. The value of determinant is unchanged if its rows are interchanged by its columns.

This means that the determinant

of matrix A is equal to the value of the determinant of the transpose matrix of A

$$|A^T| = |A|$$

5- If multiple of any row or columns of a determinant is added to any other row or columns, then the value of the determinant is unchanged.

Examples :

$$D = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$D = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 0$$

property 1a

$$D = \begin{vmatrix} 2 & 4 & 5 \\ 2 & 4 & 5 \\ 1 & 3 & 1 \end{vmatrix} = 0$$

$$D = \begin{vmatrix} 2 & 2 & 0 \\ 3 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 0$$

property 1b

$$D = \begin{vmatrix} 5 & 3 & 6 \\ 1 & 2 & 4 \\ 0 & 4 & 8 \end{vmatrix} = 0 \qquad D = \begin{vmatrix} 5 & 7 & 2 \\ 10 & 14 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

property 1c

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 2 & 3 \\ 7 & 1 & 2 \\ 4 & 5 & 6 \end{vmatrix}$$ property 2

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} \Rightarrow 3D = \begin{vmatrix} 3 & 9 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

or $3D = \begin{vmatrix} 3 & 3 & 0 \\ 6 & 6 & 4 \\ -3 & 0 & 2 \end{vmatrix}$ property 3

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$$ property 4

$$D = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$ $R_3 = -2R_2 + R_3$

$$D = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 2 & 1 \\ -5 & -3 & 0 \end{vmatrix}$$ property 5

6- If A and B are two square matrices then:

$$|AB| = |A| * |B|$$

7- If:

$$D = \begin{vmatrix} a_{11} + b & a_{12} \\ a_{21} + C & a_{22} \end{vmatrix}$$

Then:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b & a_{12} \\ c & a_{22} \end{vmatrix}$$

Example:

Use the determinant properties to find the value of the determinant:

$$|A| = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 5 & 2 \\ 4 & 1 & 3 \end{vmatrix}$$

$$C_1 = C_1 - C_3 \Rightarrow |A| = \begin{vmatrix} 2 & 4 & 1 \\ 0 & 5 & 2 \\ 1 & 1 & 3 \end{vmatrix}$$

$$R_1 = R_1 - 2R_3 \Rightarrow |A| = \begin{vmatrix} 0 & 2 & -5 \\ 0 & 5 & 2 \\ 1 & 1 & 3 \end{vmatrix}$$

Now use the first column to find the value of $|A|$:

$$|A| = (-1)^{1+3} \cdot (1) \cdot (2 \times 2 - (-5) \times 5)$$

$$= 29$$

Inverse of a matrix:

The inverse of a number A is the number $\frac{1}{A}$ so that the product $A \cdot \frac{1}{A} = 1$.

We define the inverse of a matrix as the matrix A^{-1} so that $A \cdot A^{-1} = I$. Note that only square matrices can have inverses otherwise we could not multiply $A \cdot A^{-1}$ and $A^{-1} \cdot A$. Some square matrices do not have inverses.

To find A^{-1} of any square matrix A, we follow the steps below:

1. Find $|A|$ for the matrix A. if $|A|$ = zero then the matrix A has no inverse.

2. Find the co-factors C_{ij} of the matrix A : (C).

3. Find the transpose of the matrix C : C^T

4. The inverse of matrix A is then given by:

$$A^{-1} = \frac{C^T}{|A|}$$

Example :

Find the inverse of the matrix A given by:

$$A = \begin{pmatrix} 6 & 9 \\ 3 & 5 \end{pmatrix}$$

1. Calculate $|A|$

$$|A| = 6*5 - 9*3 = 3 \neq 0$$

2. Calculate C

$$C = \begin{pmatrix} 5 & -3 \\ -9 & 6 \end{pmatrix}$$

3. $$C^T = \begin{pmatrix} 5 & -9 \\ -3 & 6 \end{pmatrix}$$

4. $$A^{-1} = \frac{C^T}{|A|} = \begin{pmatrix} 5/3 & -3 \\ -1 & 2 \end{pmatrix}$$

Home Work: Verify that

$$A^{-1} . A = A . A^{-1} = I_2$$

Where I_2 is unit matrix of order 2.

Example: Find A^{-1} given:

$$A = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & c & a \end{pmatrix}$$

It is clear that :

$$|A| = a^2 + b^2$$

The co-factors of the elements are :

$C_{11} = a$ $\quad$ $C_{12} = 0$ $\quad$ $C_{13} = -b$

$C_{21} = -bc$ $\quad$ $C_{22} = a^2+b^2$ $\quad$ $C_{23} = -ac$

$C_{31} = b$ $\quad$ $C_{32} = 0$ $\quad$ $C_{33} = a$

$$\therefore \; C = \begin{pmatrix} a & 0 & -b \\ -bc & a^2+b^2 & -ac \\ b & 0 & a \end{pmatrix}$$

$$C^T = \begin{pmatrix} a & -bc & b \\ 0 & a^2+b^2 & 0 \\ -b & -ac & a \end{pmatrix}$$

$$\therefore \; A^{-1} = \frac{C^T}{|A|} = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} a & -bc & b \\ 0 & a^2+b^2 & 0 \\ -b & -ac & a \end{pmatrix}$$

Solving a system of linear equations :

In this context we will discuss three methods to solve a set of linear equations :

1. Using the inverse matrix :

We take a system of 3 linear equations with variables x, y and z :

$$a_{11}x + a_{12}y + a_{13}z = k_1$$

$$a_{21}x + a_{22}y + a_{23}z = k_2$$

$$a_{31}x + a_{32}y + a_{33}z = k_3$$

Using the matrix formulation we can write the coefficients a_{ij} as a matrix A and the unknowns x, y and z as a matrix r and the absolute coefficients k_i as a matrix k :

$$(A)_{3\times3} \cdot (r)_{3\times1} = (K)_{3\times1}$$

If we multiply this equation by A^{-1} then :

$$r = A^{-1} \cdot K$$

Example: solve the linear set of equations using the inverse matrix method :

$$\begin{aligned} x \quad\quad\; - z &= 5 \\ -2x + y \quad\quad &= 1 \\ x - y + 2z &= -10 \end{aligned}$$

Here :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$k = \begin{pmatrix} 5 \\ 1 \\ -10 \end{pmatrix}$$

To find r : x, y and z we must first find A^{-1}.

$$A^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$ Home Work

$$\therefore \quad r = A^{-1} . k = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

$\therefore \quad x = 1 \quad , \quad y = 3$ and $z = -4$

Home Work : Verify the solution by substituting these values in the original equations.

2. Cramer's Rule:

Consider the set of linear equations:

$$a_{11}x + a_{12}y + a_{13}z = k_1$$
$$a_{21}x + a_{22}y + a_{23}z = k_2$$
$$a_{31}x + a_{32}y + a_{33}z = k_3$$

The Cramer's rule states that the solution of the set of linear equations is given by:

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$$

$$y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|}$$

$$z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{23} & k_3 \end{vmatrix}}{|A|}$$

Example: Use Cramer's rule to solve the set of equations:

$$2x + 3y = 3$$

$$x - 2y = 5$$

Here We have : $A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$

and $K = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7$$

$$x = \frac{\begin{vmatrix} 3 & 3 \\ 5 & -2 \end{vmatrix}}{-7} = \frac{-6-15}{-7} = 3$$

$$y = \frac{\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}}{-7} = \frac{10-3}{-7} = -1$$

Note : Cramer's rule maybe used to solve n equations in n unknowns if $|A| \neq 0$

Note : If a matrix A is not square or if $|A| = 0$, then A^{-1} is not exist and A is called singular matrix.

3- Row reduction method (Gaussian elimination) :

As an example of row reduction method in solving a set of linear equations we will solve the following set of equations :

$$\begin{aligned} 2x \quad\quad\quad\; - z &= 2 \\ 6x + 5y + 3z &= 7 \\ 2x - y \quad\quad\quad &= 4 \end{aligned}$$

First, we arrange the variables and constants as above and then write the matrix corresponding to the coefficients and constants as below:

$$\begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix} \qquad \begin{aligned} R_2 &= R_2 - 3R_1 \\ R_3 &= R_3 - R_1 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix} \qquad R_2 \longleftrightarrow R_3$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 6 & 1 \end{pmatrix} \qquad R_3 = R_3 + 5R_2$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{pmatrix} \qquad R_3 = R_3 / 11$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We have got upper triangle matrix.

Now we start the back substitution procedure :

From the third row :

$z = 1$

From the second row :

$-y + z = 2 \Rightarrow -y + 1 = 2 \Rightarrow y = -1$

From the first row :

$2x - z = 2 \Rightarrow 2x - 1 = 2 \Rightarrow x = \frac{3}{2}$

$\therefore$ The solution is : $\left(\frac{3}{2}, -1, 1\right)$

Note : The allowed operations on the matrix :

1. Interchange two rows
2. Multiply (or divide) a row by a nonzero constant
3. Add a multiple of one row to another, this includes subtracting

Example : Using row reduction method solve the following set of equations :

$$\begin{aligned} 10y - z + w &= 10 \\ 2x - 2y - 4z &= -3 \\ 4x + 2y + 4w &= 5 \\ 3x + 2y + 3w &= 4 \end{aligned}$$

The required matrix is :

$$\begin{pmatrix} 0 & 10 & -1 & 1 & 10 \\ 2 & -2 & -4 & 0 & -3 \\ 4 & 2 & 0 & 4 & 5 \\ 3 & 2 & 0 & 3 & 4 \end{pmatrix} \quad R_3 = R_3 - R_4$$

$$\Rightarrow \begin{pmatrix} 0 & 10 & -1 & 1 & 10 \\ 2 & -2 & -4 & 0 & -3 \\ 1 & 0 & 0 & 1 & 1 \\ 3 & 2 & 0 & 3 & 4 \end{pmatrix} \quad R_1 \longleftrightarrow R_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 2 & -2 & -4 & 0 & -3 \\ 0 & 10 & -1 & 1 & 10 \\ 3 & 2 & 0 & 3 & 4 \end{pmatrix} \quad \begin{aligned} R_2 &= R_2 - 2R_1 \\ R_4 &= R_4 - 3R_1 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & -2 & -4 & -2 & -5 \\ 0 & 10 & -1 & 1 & 10 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix} \quad \begin{aligned} R_3 &= R_3 + 5R_2 \\ R_4 &= R_4 + R_2 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & -2 & -4 & -2 & -5 \\ 0 & 0 & -21 & -9 & -15 \\ 0 & 0 & -4 & -2 & -4 \end{pmatrix} \quad \begin{aligned} R_2 &= R_2 / -1 \\ R_3 &= R_3 / -3 \\ R_4 &= R_4 / -2 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 4 & 2 & 5 \\ 0 & 0 & 7 & 3 & 5 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix} \qquad R_3 = R_3 - 3R_4$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 4 & 2 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix} \qquad R_4 = R_4 - 2R_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 4 & 2 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

start the back substitution procedure:

From the fourth row:
$w = 4$

From the third row:
$z = -1$

From the second row:
$2y + 4z + 2w = 5 \Rightarrow y = \frac{1}{2}$

From the first row:
$x = 1 - w = -3$

$\therefore$ The solution is:

$$(x, y, z, w) = \left(-3, \frac{1}{2}, -1, 4\right)$$

The rank of a Matrix

The maximum of linearly independent rows in a matrix A is called the row rank of A, and the maximum number of linearly independent columns in A is called the column rank of A.

If A is an $m \times n$ matrix then:

$$\text{row rank of } A \leq m$$

$$\text{column rank of } A \leq n$$

For any matrix A,

$$\text{The row rank of } A = \text{The column rank of } A.$$

Because of this fact, there is no reason to distinguish between row rank and column rank; the common value is simply called the rank of the matrix.

Therefore, if A is $m \times n$ matrix then:

$$\text{rank}\left(A_{m \times n}\right) \leq \min(m, n)$$

Where $\min(m, n)$ denotes the smaller of the two numbers m and n or their common value if $m = n$.

For example, the rank of 3x5 matrix can be no more than 3, and the rank of a 4x2 matrix can be no more than 2.

A row or a column is considered to be independent if it satisfies the following conditions :

1. A row/column should have at least one non-zero element.

2. A row/column should not be identical to another row/column.

3. A row/column should not be multiple of another row/column.

4. A row/column should not be a linear combination of another row/column.

For example, the rank of the below matrix would be 1 as the second row is multiple of the first and the third row does not have a non-zero element :

$$(A) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

* Rank of 2x2 matrix

The rank of a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

rank $(A) = 2$ if $|A| = ad - bc \neq 0$

rank $(A) = 1$ if $|A| = 0$ but $A \neq 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

rank $(A) = 0$ if $A = 0$

* How to compute rank (A) for an $m \times n$ matrix A?

The process by which the rank of a matrix is determined can be illustrated by the following example:

Suppose A is the 4x4 matrix:

$$\begin{pmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{pmatrix}$$

The four row vectors of this matrix are:

$$R_1 = (1 \quad -2 \quad 0 \quad 4)$$
$$R_2 = (3 \quad 1 \quad 1 \quad 0)$$
$$R_3 = (-1 \quad -5 \quad -1 \quad 8)$$
$$R_4 = (3 \quad 8 \quad 2 \quad -12)$$

These vectors are not independent since for example :

$$R_3 = 2R_1 - R_2 \qquad \ldots 1$$

and $R_4 = -3R_1 + 2R_2 \qquad \ldots 2$

The fact that the vectors R_3 and R_4 can be written as linear combinations of the other two (R_1 and R_2 which are independent) means that the maximum number of independent rows is 2. Thus, the rank of this matrix is 2.

The equations 1 and 2 can be rewritten as follows:

$$-2R_1 + R_2 + R_3 = 0 \qquad \ldots 3$$

$$3R_1 - 2R_2 + R_4 = 0 \qquad \ldots 4$$

Equation (3) implies that if -2 times that of first row is added to the third and then the second row is added to the (new) third row, the third row will become 0, (a row of zeros).

Equation (4) says that similar operations performed on the fourth row can produce a row of zeros there also.

If after these operations are completed, -3 times the first row is then added to the second row (to clear out all entries below the entry $a_{11} = 1$ in the first column), these elementary row operations reduce the matrix A to a form called : echelon form:

$$\begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The fact that there are exactly 2 nonzero rows in the echelon matrix indicates that the maximum number of linearly independent rows is 2; hence the rank$(A) = 2$.

In general, to compute the rank of a matrix, perform elementary row operations until the matrix is left in echelon form; the number of non-zero rows remaining in the reduced matrix is the rank.

Example : Find the rank of the matrix :

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

Because the matrix is 4×3, its rank can be no greater than 3.

Perform the following row operations:

$R_1 \leftrightarrow R_2$ $\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$

$R_2 = R_2 - 2R_1$
$R_4 = R_4 - R_1$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$

$R_3 = R_3 + 2R_2$
$R_4 = R_4 + R_2$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

$(-1) R_2$
$R_4 = R_4 - 4R_3$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Since there are 3 nonzero rows remaining in this echelon form, the rank = 3

Example: Determine the rank of the 4x4 matrix:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

perform the following elementary row operations:

$R_2 = R_2 + R_1$
$R_3 = R_3 - R_1$
$R_4 = R_4 + R_1$

we get: $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

since only 1 nonzero row remains; the rank = 1.

H.W.

* find the rank of the matrices:

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

* Rank and linear systems:

A linear system may behave in any one of three posible ways:

1. The system has infinitely many solutions.
2. The system has a single unique solution.
3. The system has no solution.

A linear system is said to be consistent if it has a solution, and inconsistent otherwise.

It is possible to derive a contradiction from the equations, that may always be written such as the statement $0 = 1$.

For example, the equations:

$$3x + 2y = 6$$

and

$$3x + 2y = 12$$

are inconsistent. In fact, by subtracting the first equation from the second one and multiplying both sides of the result by $\frac{1}{6}$, we get $0=1$.

Theorem :

If $Ax = b$ is a linear system of n unknowns, and $A_b = (A|b)$ is the augmented matrix then :

1. The linear system is consistent if and only if :

$$\text{rank}(A_b) = \text{rank}(A).$$

2. If the linear system is consistent, then it has a unique solution if and only if :

$$\text{rank}(A) = n$$

3. If $\text{rank}(A) < n$, then the system has $n - \text{rank}(A)$ free variables.

Example : Is the following linear system consistent ? Does it have a unique solution ?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= 1 \\ 4x_1 \qquad\quad + 2x_3 &= 2 \\ 6x_2 - 3x_3 &= 4 \end{aligned}$$

We have :

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -1 \end{pmatrix} \qquad A_b = \begin{pmatrix} 2 & 2 & -1 & 1 \\ 4 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \end{pmatrix}$$

$\text{rank}(A) = 3$
$\text{rank}(A_b) = 3$

$\therefore$ The system is consistent.

$$n - \text{rank}(A) = 3 - 3 = 0$$

So the system has a unique solution.

Example :

Find the rank of the linear system :

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + 3x_3 + 3x_4 &= 0 \end{aligned}$$

Note that the system is consistent since it has the trivial solution $X = 0$

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix}$$

We can find that rank(A) = 2 (H.W.)

∴ The system has n − rank(A) free variables = 4 − 2 = 2

We can verify that :

$$x_1 - x_2 = -2x_3 - 3x_4$$
$$x_2 = -x_3$$

This gives

$$x_1 = -3x_3 - 3x_4$$
$$x_2 = -x_3$$
$x_3 =$ free variable (can take)
$x_4 =$ free variable (any value)

For example : put $x_3 = 1$
$x_4 = 1$
then $x_2 = -1$
$x_1 = -6$

This set of values can verify the original set of equations.

* Eigenvalues and Eigenvectors:

We say λ is an eigenvalue of a square matrix A if:

$$Ax = \lambda x$$

For some $x \neq 0$. The vector x is called an eigenvector of A associated with the eigen value λ.

Note that if x is an eigenvector. then any multiple αx is also an eigenvector.

For an $n \times n$ matrix A, solving $Ax = \lambda x$ for a vector $x \neq 0$ is equivalent to solving the homogeneous linear system:

$$(A - \lambda I)x = 0$$

This has a nonzero solution if and only if:

$$|A - \lambda I| = 0$$

$$f_A(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

We can expand this determinant to get a polynomial of λ.

$f_A(\lambda)$ is the characteristic polynomial of A and:

$$f_A(\lambda) = 0$$

is called the characteristic equation of A.

1. $f_A(\lambda)$ is a polynomial of degree n.

2. The matrix A has at least one eigenvalue.

3. A has at most n distinct eigenvalues.

The multiplicity of λ as a root of $f_A(\lambda) = 0$ is called the algebric multiplicity of λ. The number of independent eigenvectors associated with λ is called geometric multiplicity of λ.

Examples :

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

has the eigen value $\lambda = 1$ with both algebric and geometric multiplicity equal to 2.

The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has the eigenvalue $\lambda = 1$ with algebric multiplicity 2 and geometric multiplicity 1.

Example: Find the eigenvalues of the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

Solution:

We have to solve the equation:

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)(3-\lambda) - 8 = 0$$

$$\therefore \lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$\therefore$ The matrix A has two eigenvalues 5 and -1.

Example : Find the eigenvalues of the matrix :

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Solution :

We have to solve the equation :

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 0 & 2-\lambda & 3 & 4 & 5 \\ 0 & 0 & 3-\lambda & 4 & 5 \\ 0 & 0 & 0 & 4-\lambda & 5 \\ 0 & 0 & 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)(5-\lambda) = 0$$

$\therefore$ There are five eigenvalues 1, 2, 3, 4 and 5 for matrix A.

Fact : The eigenvalues of a triangular matrix are its diagonal elements.

Example : Find $f_A(\lambda)$ for the 2×2 matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f_A(\lambda) = |A - \lambda I_2|$$

$$= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

The constant term is $|A|$. why?

The constant term is $f_A(0)$

$$= |A - 0 I_2| = |A|.$$

The coefficient of λ is the negative of the sum of the diagonal entries of A.

Definition : Trace

The sum of the diagonal entries of an $n\times n$ matrix A is called the trace of A denoted by $tr(A)$.

$\therefore$ If A is a 2×2 matrix then:

$$f_A(\lambda) = \lambda^2 - tr(A)\lambda + |A|$$

For the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

we have $tr(A) = 4$

and $|A| = -5$

So that $f_A(\lambda) = \lambda^2 - 4\lambda - 5$

Example: Find the eigenvalues of

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_A(\lambda) = (\lambda-1)^3 \cdot (\lambda-2)^2$$

$\therefore$ The eigenvalues are 1 and 2.

Since 1 is a root of multiplicity 3 of the charcteristic polynomial, we say that the eigenvalue 1 has algebric multiplicity 3. Likewise, the eigenvalue 2 has algebric multiplicity 2.

Example : Find the eigenvalues with their algebric multiplicities :

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$f_A(\lambda) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)^3 + 2 - 3(\lambda-2)$$

$$= (\lambda-3)^2 . \lambda$$

The eigenvalues are 3 and 0 with algebric multiplicities 2 and 1 respectively.

Home Work :

Find the real eigenvalues and their algebric multiplicities of :

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

How to find the eigenvectors ?

The characteristic equation $|A - \lambda I| = 0$ involves only λ not x.

To find x (eigenvectors) :

for each λ solve $(A - \lambda I)x = 0$ or $Ax = \lambda x$ to find an eigenvector x.

Example : Find the eigenvalues and the eigenvectors of the matrix :

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$f_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

to find λ set $f_A(\lambda)$ to zero

$$f_A(\lambda) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$\therefore \quad \lambda^2 - 5\lambda = 0 \qquad \therefore \quad \lambda = 0$ and 5

$$\lambda_1 = 0 \qquad \lambda_2 = 5$$

Now to find the eigenvectors we must solve the equations $(A - \lambda I)x = 0$ seperately for $\lambda_1 = 0$ and $\lambda_2 = 5$

for $\lambda_1 = 0$

$$(A - 0I)x = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $y + 2z = 0$
$2y + 4z = 0$ $\Rightarrow$ $z = t$ free variable
$y = -2t$

$\therefore$ $X = \begin{pmatrix} t \\ -2t \end{pmatrix}$ where t is any number.

for $\lambda_2 = 5$

$$(A - 5I)x = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-4y + 2z = 0$
$2y - z = 0$ $\Rightarrow$ $y = t$ free variable
$z = 2t$

$\therefore$ $X = \begin{pmatrix} t \\ 2t \end{pmatrix}$ where t is any number.

* Some facts about the eigenvalues :

1. If we add a row of A to another row or exchange rows the eigenvalues usually change.

2. The triangular U has its eigenvalues sitting along the diagonal (pivots).

3- The product of the eigenvalues equals the determinant.

4. The sum of the eigen values equals the sum of the diagonal entries (trace).

Examples:

* $B = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = 1$

$$\lambda_1 . \lambda_2 = |B| = 0$$

$$\begin{aligned} \lambda_1 + \lambda_2 = tr(B) &= B_{11} + B_{22} \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

* $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = 7$

$$\lambda_1 . \lambda_2 = |A| = 0$$

$$\begin{aligned} \lambda_1 + \lambda_2 = 0 + 7 &= A_{11} + A_{22} \\ &= 1 + 6 \\ &= 7 \end{aligned}$$

5- The eigenvalues might not be real numbers.

Example: The 90^0 rotation

$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has no real eigenvalues

The eigenvalues of Q are:

$\lambda_1 = i$ and $\lambda_2 = -i$

$\lambda_1 + \lambda_2 = \text{trace} = 0$

$\lambda_1 \cdot \lambda_2 = |Q| = 1$

6 - The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1} with the same eigenvectors.

This means that:

if $Ax = \lambda x$

Then $A^2 x = \lambda^2 x$

and $A^{-1} x = \lambda^{-1} x$

Home Work:

1. Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

2. Find the eigenvalues and eigenvectors of the 3X3 matrix A:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

3. Find the eigenvalues and eigenvectors of these two matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \qquad A + I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

4. Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace:

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

5. Find the eigenvalues of A and B and $A+B$:

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

(Chapter 4)

(Complex Algebra)

The number system as we know it today is a result of gradual development as indicated in the following list:

1. Natural numbers: 1, 2, 3, 4, ... also called positive integers were first used in counting. If a and b are natural numbers, the sum $a+b$ and product $a * b$ are also natural numbers. For this reason the set of natural numbers is said to be closed under the operations of addition and multiplication.

2. Negative integers and Zero: Denoted by $-1, -2, -3, \ldots$ and 0. They arose to permit solutions of equations such as $x + b = a$ where a and b are natural numbers such that $b \geq a$. This leads to the operation of subtraction $x = a - b$. The set of positive and negative integers and zero is called the set of integers and is closed under the operations of addition, multiplication and subtraction.

3- Rational numbers : such as $\frac{3}{4}$, $-\frac{8}{3}$, ... arose to permit solutions of equations such as $bx = a$ for all integers of a and b where $b \neq 0$. This leads to the operation of division $x = a/b$ called the quotient of a and b where a is the numerator and b is the denominator. The set of integers is a part or subset of the rational numbers since integers correspond to rational numbers a/b where $b = 1$.
The set of rational numbers is closed under the operations of addition, subtraction, multiplication and division so long as division by zero is excluded.

4- Irrational numbers : such as $\sqrt{2} = 1.41423...$ and $\pi = 3.14159....$ are numbers which are not rational (cannot be expressed as a/b where a and b are integers and $b \neq 0$). The set of rational and irrational numbers is called the set of real numbers.

* The Complex number system

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

We can consider a complex number as having the form $a + bi$ where a and b are real numbers and i which is called the imaginary unit has the property $i^2 = -1$.

If $z = a + bi$, then a is called the real part of z and b is called the imaginary part of z and are denoted by $Re(z)$ and $Im(z)$ respectively.

The symbol z which can stand for any of a set of complex numbers is called a complex variable.

Two complex numbers $a+bi$ and $c+di$ are equal if and only if $a=c$ and $b=d$.

We can consider real numbers as a subset of the set of complex numbers with $b=0$. Thus the complex numbers $0+0i$ and $-3+0i$ represent the real numbers 0 and -3 respectively.

If $a=0$, the complex number $0+bi$ or bi is called pure imaginary number.

The complex conjugate of a complex number $a+bi$ is $a-bi$.

The complex conjugate of a complex number z is often denoted by $\bar{z}$ or z^*.

* Fundamental Operations with Complex numbers :

1. Addition :

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

2. Subtraction :

$$(a+bi)-(c+di) = (a-c)+(b-d)i$$

3. Multiplication :

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

4. Division :

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$

* Absolute Value :

The absolute value or modulus of a complex number $a+bi$ is defined as:

$$|a+ib| = \sqrt{a^2+b^2}$$

Example : $|-4+2i| = \sqrt{20}$

If $z_1, z_2, z_3 \dots z_m$ are complex numbers, the following properties hold :

1. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ (prove it)

or $|z_1 \cdot z_2 \cdots z_m| = |z_1| \cdot |z_2| \cdots |z_m|$

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$ (prove it)

3. $|z_1 + z_2| \leq |z_1| + |z_2|$ (prove it)

or $|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$

4. $|z_1 + z_2| \geq |z_1| - |z_2|$ (prove it)

or $|z_1 - z_2| \geq |z_1| - |z_2|$ (prove it)

* Sometimes it is desirable to define a complex number as an ordered pair (a,b) of real numbers a and b subject to certain operation equivalent to the complex number. These definitions are as follows:

1. Equality $(a,b) = (c,d)$ if and only if $a=c$, $b=d$
2. Sum $(a,b) + (c,d) = (a+c, b+d)$
3. product $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$

$$m(a,b) = (ma, mb)$$

We can show that $(a,b) = a(1,0) + b(0,1)$ where $a+bi$ is a complex number and i is the symbol for $(0,1)$ and has the property that $i^2 = (0,1)(0,1) = (-1,0)$ which is equivalent to the real number -1. $(1,0)$ can be considered equivalent to the real number 1. The ordered pair $(0,0)$ corresponds to the real number 0.

We can prove that if z_1, z_2, z_3 belong to the set S of complex numbers, then:

1. $z_1 + z_2$ and $z_1 . z_2$ belong to S

   Closure Law

2. $z_1 + z_2 = z_2 + z_1$

   Commutative law of addition

3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

   Associative law of addition

4. $z_1 . z_2 = z_2 . z_1$

   Commutative law of multiplication

5. $z_1 . (z_2 . z_3) = (z_1 . z_2) . z_3$

   Associative law of multiplication

6. $z_1 . (z_2 + z_3) = z_1 . z_2 + z_1 . z_3$

   Distributive law

7. $Z_1 + 0 = 0 + Z_1 = Z_1$

0 is called the identity with respect to addition

$Z_1 . 1 = 1 . Z_1 = Z_1$

1 is called the identity with respect to multiplication

8. For any complex number Z_1 there is a unique number Z in S such that:

$$Z + Z_1 = 0$$

Z is called the inverse of Z_1 with respect to addition and is denoted by $-Z_1$

9. For any $Z_1 \neq 0$ there is a unique number Z in S such that:

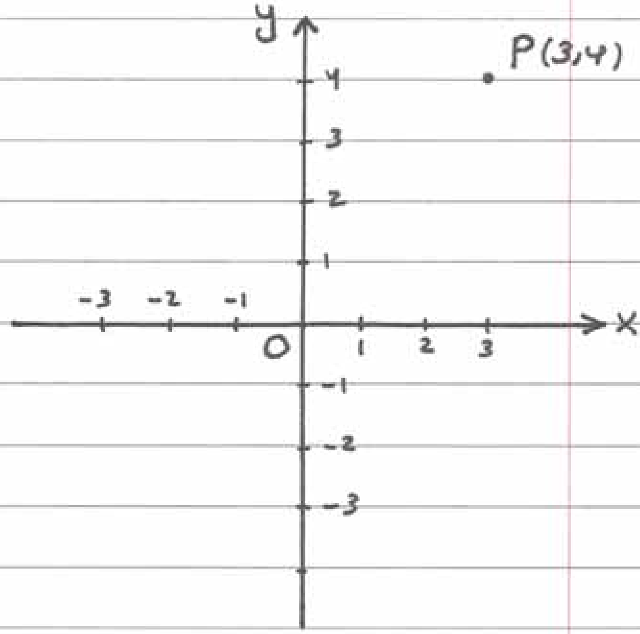
$$Z_1 . Z = Z . Z_1 = 1$$

Z is called the inverse of Z_1 with respect to multiplication and is denoted by Z_1^{-1} or $1/Z_1$.

* Graphical Representation of Complex Numbers:

Since a complex number $x+iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in any xy plane called the complex plane.

The complex number represented by P for example could then be read either (3,4) or $3+4i$. To each complex number there corresponds one and only one point in the plane and the opposite must be true also. Because of this we often refer to the complex number Z as the point Z.



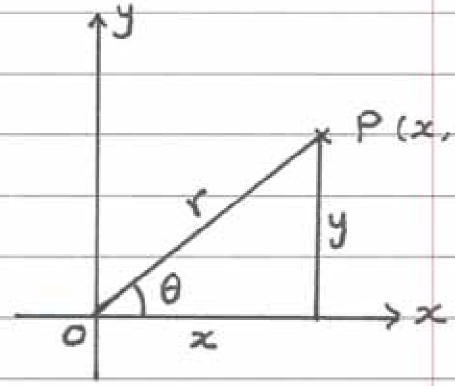
Here we refer to the x and y axes as the real and imaginary axes respectively and to the complex plane as the Z plane.

The distance between two points $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ in the complex plane is given by:

$$|Z_1 - Z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

* Polar Form of Complex Numbers:

IF p is a point in the complex plane corresponding to the complex number (x, y) or $(x+iy)$ then we can write:



$$x = r\cos\theta$$
$$y = r\sin\theta$$

Where $r = \sqrt{x^2+y^2}$

$$= |x+iy|$$

is called the modulus value of $z = x+iy$ denoted by $\text{mod } z$ or $|z|$ and θ is called the argument of $z = x+iy$ denoted by $\arg z$.

It follows that $z = x+iy$

$$= r(\cos\theta + i\sin\theta)$$

which is called the polar form of the complex number and r, θ are called polar coordinates.

* De Moivre Theorem:

If $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

We can show that:

$$z_1 . z_2 = r_1 . r_2 \Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big)$$

and $\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\cdot\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

prove it (Home Work).

In general :

$$Z_1 \cdot Z_2 \cdot \ldots Z_n = r_1 \cdot r_2 \cdot r_3 \cdots r_n \Big(\cos(\theta_1+\theta_2+\cdots+\theta_n)$$
$$+ i\sin(\theta_1+\theta_2+\cdots+\theta_n)\Big)$$

and if $Z_1 = Z_2 = \ldots = Z_n = Z$

$$\therefore \; Z^n = r^n \cdot (\cos n\theta + i\sin n\theta)$$

Which is often called De Moivre's theorem.
prove it (Home work)

* Roots of Complex Numbers :

A number ω is called the n^{th} root of a complex number z if $\omega^n = z$ and we write $\omega = z^{1/n}$.

From De Moivre's theorem we can show that if n is a positive integer.

$$z^{1/n} = \left\{ r(\cos\theta + i\sin\theta)\right\}^{1/n}$$
$$= r^{1/n} \cdot \left\{ \cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}$$

$$k = 0, 1, 2, \ldots, n-1$$

from which it follows that there are n different values for $z^{1/n}$, i.e. n different n^{th} roots of z, provided $z \neq 0$.

Euler's Formula :

By assuming that the infinite series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ of elementary calculus holds when $x = i\theta$ we can arrive at the result:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Which is called Euler's formula.

In general we define :

$$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$$

$$= e^{x} \cdot (\cos y + i \sin y)$$

In special case where $y = 0$ this reduces to e^x.

De Moivre's theorem can be reduced to :

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Logarithm of complex numbers:

If $z = a + ib$ is a complex number then we can write it in the polar form considering that the phase angle can take many values as follows:

$$z = r e^{i(\theta + 2\pi m)} \qquad m = 0, \mp 1, \mp 2, \ldots$$

$$\therefore \ln z = \ln \left(r e^{i(\theta + 2\pi m)} \right)$$

And from this equation we can see that $\ln z$ has more than one value depending on the value of m. This, of course disagree with the definition of function which states that it should be unique valued.

Hence, $\ln z = \ln r + i\theta$
And it is called the main branch of the logarithm

Example: Find the logarithm of the complex number $z = -5 - 5i$

$$r^2 = x^2 + y^2 = 50 \qquad \therefore r = 5\sqrt{2}$$

$r\cos\theta = -5$

$r\sin\theta = -5$ $\qquad$ $\cos\theta$ is negative

$\therefore \tan\theta = 1$ $\qquad$ $\sin\theta$ is negative

$\tan\theta$ is positive $= 1$

$$\therefore \theta = \frac{5\pi}{4}$$

$\therefore$ The polar form of z is:

$$z = 5\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$= 5\sqrt{2}\, e^{i\left(\frac{5\pi}{4}\right)}$$

$$\therefore \ln z = \ln 5\sqrt{2} + i\,\frac{5\pi}{4}$$

* Complex powers of real and complex numbers:

The number $z_1^{z_2}$ can be defined as:

$$z_1^{z_2} = e^{z_2 \cdot \ln z_1}$$

Example: Evaluate the number 2^i

$$z = 2^i = e^{i\ln 2}$$

Recall the formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore z = \cos(\ln 2) + i\sin(\ln 2)$$
$$= 0.77 + i\,0.639$$

Example: Evaluate the number i^{2i}

$$i^{2i} = e^{2i \cdot \ln i}$$

$$i = e^{i\left(\frac{\pi}{2} + 2\pi m\right)}$$

$$\therefore \ln i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi m\right)$$
$$= i\left(\frac{\pi}{2} + 2\pi m\right)$$

$$\therefore i^{2i} = e^{2i\left(i\frac{\pi}{2} + 2\pi m i\right)}$$

$$= e^{-\pi - 4\pi m} \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$\therefore i^{2i} = e^{-\pi}, e^{-5\pi}, e^{3\pi}, e^{-9\pi}, e^{7\pi}, \ldots$$

$e^{-\pi}$ is the main branch value.

Note that all the values of i^{2i} are real.

Example: Evaluate 10^{iz} where $z = 3+2i$

$$10^{iz} = e^{iz \ln 10} = e^{i(3\ln 10 + 2i\ln 10)}$$

$$= e^{-2\ln 10} \cdot \left[\cos(3\ln 10) + i \sin(3\ln 10)\right]$$

$$= 0.01 * \left[0.81 + i\, 0.585\right]$$

$$= 0.008 + i\, 0.00585$$

Home Work:

1. Vibrational motion
2. problems: 1, 2, 3, 4, 5, 6, 7, 8, 12
   13, 14

Functions of Complex Variable

All the elementary functions of real variables may be extended into the complex plane by replacing the real variable x by the complex variable z.

Cauchy - Riemann Conditions:

The derivative of $f(z)$, like that of a real function is defined by:

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{(z+\delta z) - \delta z} \quad \circledast$$

$$= \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} \text{ or } f'(z)$$

For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$.

Now with $z = z_0$ some point in a plane, our requirement that the limit be independent of the direction of approach is restrictive.

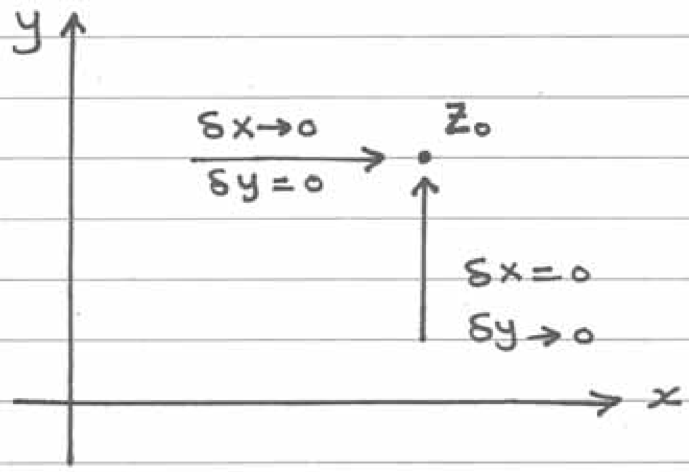
Consider increments δx and δy of the variables x and y respectively.

Then $\delta z = \delta x + i\,\delta y$

Also, $\delta f = \delta u + i\,\delta v$

so that $$\frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

Let us take the limit indicated by equation (*) by two different approaches as shown in the figure below:



The first approach : with $\delta y = 0$

we let $\delta x \to 0$ and equation (*) yields:

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x} \right)$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad \cdots (1)$$

The second approach : with $\delta x = 0$

we let $\delta y \to 0$ and equation (*) yields:

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i\,\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right)$$

$$\therefore \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \cdots (2)$$

If $\frac{df}{dz}$ exists then equations (1) and (2) must be identical.

Equating real parts to real parts and imaginary parts to imaginary parts we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are the famous Cauchy-Riemann conditions. They were discovered by Cauchy and used extensively by Riemann in his theory of analytic functions.

These Cauchy-Riemann conditions are necessary for the existence of derivative of $f(z)$. If $\frac{df}{dz}$ exists, the Cauchy-Riemann conditions must hold.

Example:

Let $f(z) = z^2$

Then the real part $u(x,y) = x^2 - y^2$ and the imaginary part $v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z) = z^2$ satisfies the Cauchy-Riemann conditions throughout the complex plane.

Since the partial derivatives are clearly continuous, we can say that $f(z) = z^2$ is analytic function.

Example :

Let $f(z) = z^*$

It is clear that $u = x$ and $v = -y$.

We can verify that :

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y}$$

The Cauchy-Riemann conditions are not satisfied and $f(z) = z^*$ is not an analytic function of z.

Example :

$$f(z) = e^{z}$$

$$= e^{x} \cdot \cos y + i \, e^{x} \cdot \sin y$$

$$u(x,y) = e^{x} \cdot \cos y$$
$$v(x,y) = e^{x} \cdot \sin y$$

$$\frac{\partial u}{\partial x} = e^{x} \cdot \cos y \qquad \frac{\partial v}{\partial y} = e^{x} \cdot \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^{x} \cdot \sin y \qquad \frac{\partial v}{\partial x} = e^{x} \cdot \sin y$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ The function $f(z) = e^{z}$ satisfies the Cauchy-Riemann conditions

$$\frac{d}{dz}(e^{z}) = \frac{\partial}{\partial x}(e^{x} \cos y) + i \frac{\partial}{\partial x}(e^{x} \sin y)$$

$$= e^{x} \cos y + i \, e^{x} \sin y$$

$$= e^{z}$$

Note that : $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Home Work

1. Let $f(z) = y - 2xy + i(-x + x^2 + y^2) + z^2$

where $z = x + iy$

Verify that Cauchy-Riemann conditions hold for all values of x and y and then find $f'(z)$.

2. If $f(z) = \cos x - i \sinh y$

prove that $f'(z)$ does not exist anywhere.

3. Show that $f(z) = (\bar{z}+1)^3 - 3\bar{z}$ is nowhere analytic.